Resilient Networks for Multi-Agent Systems based on Graph Self-Organization into Random Approximate Regular Graphs

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Abstract— **This paper proposes a distributed protocol that can self-organize a connected graph representing a network into a random approximate regular graph with an arbitrary degree, which is known to possess robustness properties against link and node failures, including also DoS network attacks. The scenario under consideration is that of an unstructured peer-to-peer network, where the agents and are allowed to close communications with their neighbors and establish new communications with two-hop neighbor, while the time-varying graph topology remains unknown. To validate the efficacy of the proposed protocol, we examine the spectral properties of the self-organizing graph, and we numerically show that they approach those of random regular graphs, particularly for large networks. We also compare the performance of the proposed protocol with the state-of-the-art, showing improvements in convergence speed and scalability, despite the absence of synchronous multi-node coordination of previous approaches in the literature.**

I. Introduction

Collaborative networks of agents are profoundly shaped by their mutual interaction pattern, which can significantly impact network performance. Graph theory offers an insightful framework for modeling such patterns, representing agents as nodes and interactions as edges connecting them. The properties of the graph play a pivotal role in modeling several key network properties, including resilience to perturbations [1], [2], controllability [3]–[5], and the feasibility of distributed algorithms [6], [7].

In many applications, multi-agent networks must deal with perturbations such as sudden disconnections of agents due to failures [8], [9] or attacks carried out by malicious agents [10]. One of the worst events that should be avoided is the disconnection of the network into two or more components, which can impede the flow of information throughout the network. To quantify the connectivity of a graph, measures have been proposed that are mostly related to the number of nodes and edges that need to be removed to disconnect the graph. Some well-established measures include algebraic connectivity and the Fiedler eigenvector [11], [12], the Kirchoff index [13], [14], and the edge/node expansion ratio [15], [16]. These measures reflect the quality of connectivity of a graph and have been employed to characterize its robustness and synchronizability [17].

Algorithms aimed at enhancing graph connectivity based on diverse connectivity metrics have garnered significant attention [18]–[21]. The trivial attempt of adding more edges to the graph to increase the connectivity may not be feasible or convenient in many scenarios, such as the case when edges represent physical communication channels [22], in which case the existence of an edge corresponds to an economic burden. Thus, at an equivalent level of connectivity, a sparse graph may be preferable to a dense one. An intriguing class of graphs that attains high connectivity levels across various measures while also maintaining significant levels of sparsity is represented by random regular graphs [23]. A regular graph is such that all nodes have the same number $k \in \mathbb{N}$ of edges, thus called k-regular; for a graph to be k-regular, the product nk must be even, ensuring that the total number of edges is an integer. Such graphs are also random if they are selected at random, with uniform probability, from the set of all regular graphs with the same number of nodes and edges.

The **problem of interest** of this study is to increase network resilience to disconnection, resulting from the loss of nodes or links, in a distributed manner, leveraging only locally available information. The proposed approach differs from the series of works by Yazıcıoğlu, Egerstedt, and Shamma [23]–[25] in that it allows for arbitrary selection of regularity degree $k \in \mathbb{N}$ by loosening the standard regularity notion. Specifically, we introduce the concept of random Δ-*approximate* k-regular graphs, where nodes have a variable number of edges within the range $[k, k+\Delta]$ where $\Delta \in \mathbb{N}$. These graphs exhibit similar robustness and spectral properties to regular graphs.

The **main contribution** of this paper is the development of a novel distributed protocol that enables the self-organization of a connected graph into what we term as a random 2-approximate k-regular graph in multi-agent networks, offering the following set of features:

- it allows for the arbitrary choice of the regularity degree $k \geq 2$, regardless on the specific initial configuration of the graphs. This fact allows to employ the proposed protocol to control the algebraic connectivity of any graph such that is greater than a certain desired value

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$\lambda_{2,lb}$ that is a function k, namely $\lambda_{2,lb} = k - 2\sqrt{k-1}$.
- it solely relies on autonomous decisions made by each agent, thus not requiring any coordination between them. This property significantly reduces the convergence time of the proposed protocol w.r.t. other protocols in the current literature exploiting coordinated decision-making between neighboring agents;
- it can be employed also in the framework of open multi-agent systems, where agents may leave or join the network as time goes by. This flexibility allows to maintain well connected a network where node and link failures may arise due to, e.g., malfunctioning or malicious attacks.

We numerically show that the spectral distribution of the random 2-approximate k-regular graph obtained by the proposed self-organization protocol resembles that of random k-regular graphs. In particular, the self-organized graphs possess high algebraic connectivity for a given number of edges (at most $(k+2)n/2$ where n is the number of nodes), resulting in enhanced resilience to node and link failures.

Structure of the paper: Section II presents some notations and preliminaries. Section III outlines the proposed distributed algorithm for solving the formalized problem along with a numerical analysis showing that the spectral distribution of the random 2-approximate k-regular graph proposed in this paper approaches that of random k-regular graphs. Numerical simulations showcasing the effectiveness and superiority of the proposed algorithm compared to the state-of-the-art in Section IV. The paper concludes with a summary of the work and future directions in Section V.

II. Notation and Preliminaries

We denote with $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers respectively. Moreover, we denote with $\mathbb{R}_+$ and $\mathbb{N}_+$ their restriction to strictly positive numbers.

A. Networks and graph theory

We consider complex systems consisting of multiple interacting agents and refer to them as *Multi-Agent Systems* (MASs). Let $n \in \mathbb{N}$ be the number of agents, $\mathcal{V} = \{1, 2, \dots, n\}$ be the set representing the agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ be a subset of all the pairs among agents representing their local interconnections. These sets uniquely define the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of *nodes* and $\mathcal{E}$ the set of *edges* or *links*. Interactions among agents are assumed to be *undirected*, i.e., (i, j) and (j, i) denote the same link between agents $i, j \in \mathcal{V}$.

A path between two nodes $i, j \in \mathcal{V}$ is a series of edges $(i, p), (p, q), ..., (r, s), (s, j)$, where each pair of consecutive edges in the sequence shares a common node. An undirected graph $\mathcal{G}$ is said to be *connected* if there exists at least one such path connecting every pair of nodes. When nodes $i, j \in \mathcal{V}$ share a link $(i, j) \in \mathcal{E}$ are said to be *neighbors*; all neighbors of a given node $i \in \mathcal{V}$ are included in the set $\mathcal{N}_i = \{j \in \mathcal{V} \setminus \{i\} : (i, j) \in \mathcal{E}\}$. The neighbors of the neighbors are said to be 2*-hop neighbors*, which are included in the set $\mathcal{N}_i^2 = \{k \in \mathcal{V} \setminus \mathcal{N}_i : (k, j) \in \mathcal{E} \text{ with } j \in \mathcal{N}_i, \}$. The number of neighbors of a given node $i \in \mathcal{V}$ is called the *degree* and is denoted by $d_i = |\mathcal{N}_i|$, where $|\cdot|$ denotes the cardinality of a set. Consequently, the minimum and maximum degrees of the nodes in the graph are

$$d_{\text{MIN}}(\mathcal{G}) = \min_{i \in \mathcal{V}} d_i, \quad d_{\text{MAX}}(\mathcal{G}) = \max_{i \in \mathcal{V}} d_i,$$

respectively. The *Laplacian matrix* of graph $\mathcal{G}$ is defined as $L_n = D - A_n \in \mathbb{N}^{n \times n}$, where $D = \{d_{ij}\} \in \mathbb{N}^{n \times n}$ is the *degree matrix*, a diagonal matrix with diagonal entries equal to the degrees of the nodes ($d_{ii} = d_i$) and whose off-diagonal entries are zero ($d_{ij} = 0$ for $i \neq j$); $A_n\{a_{ij}\} \in \{0, 1\}^{n \times n}$ is the *adjacency matrix* such that $a_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The eigenvalues of the Laplacian matrix L_n and of the adjacency matrix A_n are denoted by λ_i and μ_i for $i = 1, \dots, n$, respectively; we assume that they are always sorted in ascending order, i.e., $\mu_i \leq \mu_{i+1}$ and $\lambda_i \leq \lambda_{i+1}$ for $i = 1, \dots, n-1$. Finally, we recall that the second smallest eigenvalue λ_2 of the Laplacian matrix L_n is known as the *algebraic connectivity* of the graph.

B. Link ownership and random approximate regular graphs

We consider a scenario MAS wherein the agents are allowed to create new links with other agents – which essentially entails adding new edges to the network graph – becoming in this way the *owner* of the link. This ownership status gives them the exclusive right to delete links, but only the ones they created, thereby avoiding conflicts such as when two adjacent agents disagree on whether to remove a link. We model this scenario by introducing a directed *ownership graph* $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$ such that if $(i, j) \in \mathcal{E}_d$ then $(j, i) \notin \mathcal{E}_d$ and agent i is the owner of the link with agent j. Thus, the graph $\mathcal{G}$ describing the interconnections among the agents can be viewed as the undirected version of $\mathcal{G}_d$. Finally, we let $\mathcal{N}_{i,own} = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}_d\}$ denote the set of links owned by agent i.

In such a dynamic scenario where the graph changes over time according to the autonomous actions taken by the agents becomes of paramount importance to keep the graph connected with high levels of connectivity, thus promoting robustness and resilience properties. A class of graphs that plays a pivotal role in this direction is that of random regular graphs.

Definition 1: *A graph with n nodes is said to be "k-regular" if each node has degree k such that the product nk is even, where $n, k \geq 2$ are integers.*

When considering arbitrary values of the product nk, ensuring that all nodes have an equal degree k may not always be achievable; indeed, n and k must be such that the product nk is even. In pursuit of practical implementation, especially when the number of agents n is unknown (or time-varying) and the regularity degree is arbitrary, we introduce the notion of *approximate regular graphs*. These graphs aim to approximate the desired regularity by allowing some flexibility in the degree distribution while still maintaining a relatively uniform connectivity pattern.

Definition 2: *A graph with n nodes is said to be "Δ-approximate k-regular" if each node has the degree within $[k, k+\Delta]$, where $n, k \geq 2$ and $\Delta \geq 0$ are integers.*

Definition 3: *A Δ-approximate k-regular graph is said to be "random" if it is selected uniformly at random from all Δ-approximate k-regular graphs with the same number of nodes, where $n, k \geq 2$ and $\Delta \geq 0$ are integers.*

III. Proposed Distributed Algorithm

We detail the proposed algorithm in Algorithm 1. At each time step, each agent $i \in \mathcal{V}$ is randomly activated and locally modifies the network topology according to its own degree d_i and the desired integer degree $k \geq 2$ of the final 2-approximate k-regular graph. It is worth noting that not all agents are necessarily activated sequentially at each time step. Instead, each node is activated at random with equal probability during the procedure. The actions performed by each agent to locally modify the network topology are completely independent and do not involve any coordination with other agents. We provide a detailed explanation of the rules governing these actions as below.

[Rule (A): add edges while $d_i \leq k-1$] Agent i picks at random neighbors $j \in \mathcal{N}_i$ until it finds one with a neighbor $s \in \mathcal{N}_j$ that is not i itself or a neighbor of i. Agent i then picks at random such agent s among those with a minimum degree and opens the communication with agent s, i.e., edge (i,s) is added to $\mathcal{E}$. This process is repeated until d_i is greater than $k-1$. Outcome: The degree of agent i satisfies $d_i \geq k$.

[Rule (R): remove edges while $d_i \geq k+1$ and there is $j \in \mathcal{N}_{i,own}$ such that $d_j \geq k+1$] Agent i picks at random an out-neighbor $j \in \mathcal{N}_{i,own}$ with the maximum degree, such that $d_j \geq k+1$, and then closes the communication with agent j, i.e., edge (i,j) is removed from $\mathcal{E}$. Outcome: The degree of agents i and j decreases.

[Rule (M): try to move or add edges if $\mathcal{N}_{i,own} \neq \emptyset$] Agent i picks at random an out-neighbor $j \in \mathcal{N}_{i,own}$ with the maximum degree, then it picks at random a neighbor s of j that it is not i itself or a neighbor of i and such that $d_s \leq k+1$. If such an agent s does not exist no action is taken; otherwise two different actions can be taken:

- **[move an edge if $d_j \geq k+1$]** Agent i closes the communication with agent j, i.e., edge (i,j) is removed from $\mathcal{E}$, and opens the communication with agent s, i.e., edge (i,s) is added to $\mathcal{E}$. Outcome: The degree of agent i remains the same, the degree of agent j decreases by 1, the degree of agent s increases by 1.
- **[add an edge if $d_j < k+1$ and $d_i \leq k+1$]** Agent i opens the communication with agent s, i.e., edge (i,s) is added to $\mathcal{E}$. Outcome: The degree of agents i and s increases by 1.

The next Theorem 1 proves that Algorithm 1 constructs a 2-approximate regular graph, provided that the graph remains connected during its execution.

Algorithm 1: Distributed self-organization into 2-approximate k-regular graphs

```
Input: A connected graph G=(V,E), its ownership
       graph G_d(V_d,E_d), and an integer degree k>=2
Output: A random 2-approximate k-regular graph
at each step t=1,2,3,... do
    Pick at random a node i∈V
    while d_i <= k-1 do                                  // Rule (A)
        for j∈N_i do
            N_ij := {s∈N_j \ {N_i ∪ {i}}}
            if N_ij ≠ ∅ then
                N_ij^MIN := {s∈N_ij : d_s = min_{ℓ∈N_ij} d_ℓ}
                Pick at random a node s∈N_ij^MIN
                Add edge (i,s) to E
                exit for loop
    while d_i >= k+1 do                                  // Rule (R)
        N_{i,own}^≥ := {j∈N_{i,own} : d_j >= k+1}
        if N_{i,own}^≥ ≠ ∅ then
            N_{i,own}^MAX := {j∈N_{i,own}^≥ : d_j = max_{ℓ∈N_{i,own}^≥} d_ℓ}
            Pick at random a node j∈N_{i,own}^MAX
            Remove edge (i,j) from E
        else
            exit the while loop
    if N_{i,own} ≠ ∅ then                                // Rule (M)
        N_{i,own}^MAX := {j∈N_{i,own} : d_j = max_{ℓ∈N_{i,own}} d_ℓ}
        Pick at random a node j∈N_{i,own}^MAX
        N_ij^≤ := {s∈N_j \ {N_i ∪ {i}} : d_s <= k+1}
        if N_ij^≤ ≠ ∅ then
            N_ij^MIN := {s∈N_ij^≤ : d_s = min_{ℓ∈N_ij^≤} d_ℓ}
            Pick at random a node s∈N_ij^MIN
            if d_j >= k+1 then
                Add edge (i,s) to E
                Remove edge (i,j) from E
            else if d_i <= k+1 then
                Add edge (i,s) to E
```

Theorem 1: *Consider a network of $n \in \mathbb{N}$ agents interacting according to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and executing Algorithm 1 with parameter $k \geq 2$. If $\mathcal{G}$ is initially connected and remains connected thereafter, then the degree d_i of each agent $i \in \mathcal{V}$ almost surely converges to the interval $d_i \in [k, k+2]$, i.e., $\mathcal{G}$ is reorganized into a 2-approximate k-regular graph.*

Proof: A complete proof is omitted in this paper due to space limitations. However, the essential steps and logical structure can be found in Appendix A. ■

Once the graph has been reorganized into a 2-approximate k-regular graph, see Theorem 1, Algorithm 1 randomizes this graph during its execution, thus making the occurrence of any possible connected graph with the class of 2-approximate k-regular graph equally likely to occur, i.e., the graph is a random 2-approximate k-regular graph. In this preliminary work, we provide no formal proof of this fact; instead, we show by numerical simulations in the following Section III-A that the algebraic connectivity and the spectral distribution of graphs produced by Algorithm 1 approach those of random k-regular graphs for large networks, that is $\lambda_2 \geq \lambda_{2,lb} := k - 2\sqrt{k-1}$.

A last remark on Theorem 1 is about the connectivity assumption, which appears quite strong when one does not think about the fact that the algorithm is intrinsically designed to increase the algebraic connectivity above the lower bound $\lambda_{2,lb}$. In practice, since the probability of disconnection is inversely proportional to the algebraic connectivity and the chosen degree, Algorithm 1 is very unlikely to disconnect the graph if the initial algebraic connectivity is sufficiently large. When selecting from the set of random k-regular graphs, there is a possibility of choosing a graph with poor connectivity in the worst-case scenario. However, with high probability, the selected k-regular graph will exhibit high algebraic connectivity. We also note that the same assumption is made in other state-of-the-art algorithms as that in [26], against which the performance of the proposed protocol are compared in the section devoted to the numerical simulations.

A. Empirical validation of the algebraic connectivity

A k-regular graph is known to have the largest eigenvalue equal to k. All the other eigenvalues, as conjectured by Alon [16] and proved by Friedman [27], are bounded by $2\sqrt{k-1}$ with high probability when the graph is picked at random among all k-regular graphs, which allows finding a lower bound on the algebraic connectivity. We summarize these results in the following proposition.

Proposition 1: *Given a random k-regular graph, the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1} \leq \mu_n$ of its adjacency matrix A_n satisfy, with high probability (see [27, Theorem 1.1]),*

$$\mu_n = k, \qquad \max\{|\mu_{n-1}|, |\mu_1|\} \leq 2\sqrt{k-1}.$$

Thus, the second smallest eigenvalue λ_2 of the Laplacian matrix L_n, the algebraic connectivity, is lower-bounded by

$$\lambda_2 \geq \lambda_{2,lb} := k - 2\sqrt{k-1}.$$

We validate the algebraic connectivity of a graph generated by Algorithm 1 by comparing it with the lower bound for a random k-regular graph characterized in Proposition 1. We run Algorithm 1 on networks with an increasing number of agents $n \in \{100, 200, \ldots, 1000\}$ and select a degree of regularity $k = \lfloor \sqrt{n} \rfloor$ proportional to the number of nodes. Figure 1 shows the relative distance of the algebraic connectivity λ_2 from the lower bound $\lambda_{2,lb}$, i.e., $(\lambda_2 - \lambda_{2,lb})/\lambda_{2,lb}$, averaged over 100 different instances of the problem. The results show that the lower bound for random k-regular graphs provided by Proposition 1 is always achieved, and the relative distance to it with the size of the network.

B. Empirical validation of the spectral distribution

The eigenvalues μ_i of the adjacency matrix A_n associated with a random graph $\mathcal{G}$ are samples of independent, identically distributed random variables. In these terms, the *empirical spectral distribution* (ESD) $P_{A_n} : \mathbb{R} \to [0,1]$ of the matrix A_n is an estimate of the *cumulative distribution function* $P(x) : \mathbb{R} \to [0,1]$ that generated its eigenvalues,

$$P_{A_n}(x) = \frac{1}{n}|\{i : \mu_i \leq x\}|,$$

where $|\cdot|$ denotes the cardinality of a set. In simple terms, the distribution $P(x)$ is the probability that an eigenvalue takes a value less than or equal to x, and the ESD $P_{A_n}(x)$ is an approximation of this probability given the realization A_n. Moreover, by the strong law of large numbers, the ESD $P_{A_n}(x)$ almost surely converges to $P(x)$ for $n \to \infty$. Another important concept is the relative likelihood that an eigenvalue is equal to a specific value, which is given by the *probability density function* $\rho : \mathbb{R} \to \mathbb{R}$, defined by

$$\lim_{n\to\infty} P_{A_n}(x) = P(x) = \int_{-\infty}^{x} \rho(x)dx.$$

A first important characterization of the ESD of random k-regular graph has been provided by McKay in [28], building upon [29], by considering the case of a fixed degree of regularity k in the limit of $n \to \infty$.

Proposition 2: *Let A_n be the adjacency matrix of a random k-regular graph with n nodes. In the limit of $n \to \infty$, the ESD of the normalized adjacency matrix*

$$A_{n,\sigma} = \frac{1}{\sigma}A_n, \qquad \sigma = \sqrt{k-1}$$

approaches the distribution with density

$$\rho_k(x) = \begin{cases} \frac{k^2-k}{2\pi(k^2-kx^2+x^2)}\sqrt{4-x^2} & \textit{if } |x| \leq 2, \\ 0 & \textit{otherwise}. \end{cases}$$

It is easy to verify that in the limit of $k \to \infty$ the ESD of A_σ in Proposition 2 converges to a distribution with semicircle density. Following this idea, Tran, Vu, and Wang in [30, Theorem 1.5] proved the following result.

Proposition 3: *Let A_n be the adjacency matrix of a random k-regular graph with n nodes. In the limit of $k, n \to \infty$, the ESD of the normalized adjacency matrix*

$$A_{n,\sigma} = \frac{1}{\sigma}A_n, \qquad \sigma = \sqrt{k - k^2/n}$$

approaches the distribution with semicircle density

$$\rho_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & \textit{if } |x| \leq 2, \\ 0, & \textit{otherwise}. \end{cases}$$

We note that the normalization in Proposition 2 is a special case of that of Proposition 3 for $n = k^2$. The semicircle density in Proposition 3 is mostly known due to the *Wigner's semicircle law* [31], which describes the limiting spectral distribution of large random matrices with independent, identically distributed entries.

To verify the ESD of graphs generated by Algorithm 1, we compute the eigenvalues of the corresponding normalized adjacency matrix $A_{n,\sigma} = A_n/\sigma$, compute their ESDs, and compared them with that characterized in Propositions 2–3

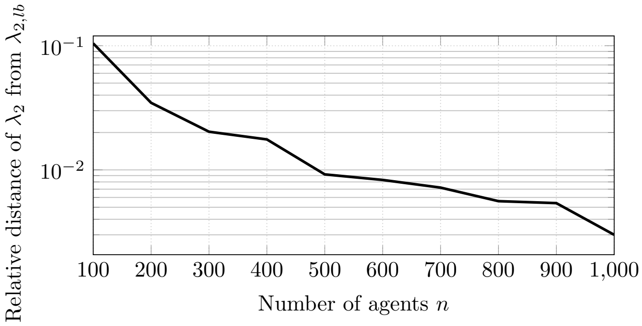

Fig. 1: Plot of $|\lambda_2 - \lambda_{2,lb}|$ for graphs generated by Algorithm 1 with $\lambda_{2,lb}$ as in Proposition 1 with $k = \lfloor\sqrt{n}\rfloor$.

for k-regular graphs. Specifically, we consider two cases:

- First, we fix the degree of regularity $k = 12$ and let the number of agents within the network increase as $n \in \{100, 250, 500, 1000\}$. The normalization factor is $\sigma = \sqrt{k-1}$ as in Proposition 2.
- Second, we fix the number of agents at $n = 1000$ and let the degree of regularity increase as $k \in \{10, 20, 30, 50\}$. The normalization factor is $\sigma = \sqrt{k - k^2/n}$ as in Proposition 3.

Figures 2–3 shows the simulation results of the above two cases, each of which has more than 10 different instances. The outcomes demonstrate that, for both cases, the ESD converges towards the distribution expected of a random k-regular graph, as detailed in Propositions 2–3. Concluding this section, we detail how the histograms in Figures 2–3 are obtained: 1) we partitioned the value range $[-2, 2]$ into 16 equal, non-overlapping segments, each with a width of 0.25; 2) we computed the eigenvalues of the normalized adjacency matrix and identified the count of eigenvalues within each specified interval; 3) we represented each segment with a bar in the histogram, where the height of the bar reflects the ratio of the number of eigenvalues in that interval to the total count. These demonstrate, from a spectral properties perspective, that the graphs generated by Algorithm 1 well approximate random k-regular graphs.

IV. Numerical simulations

We compare the performance of the proposed Algorithm 1 with respect to another algorithm proposed by some of us in [32], which, up to our knowledge, constitutes the state-of-the-art. One advantage of the proposed Algorithm 1 is that it does not require an exchange of edges between neighboring agents, each agent autonomously adds, removes, or moves edges. This autonomy is granted by the concept of "*edge ownership*", which somehow limits the operativity of the nodes in a way that any action they make is feasible, without requiring any coordination with other nodes. On the other hand, most of the current literature, as well as our previous algorithm in [32], exploits edge exchange involving synchronous multi-node coordination.

We consider a network with $n = 1000$ agents initially interacting according to a graph with average degree equal to 10, and we set the desired degree of regularity equal to $k = 50$. Figure 4 illustrates how the algebraic connectivity λ_2 (on the top), and the maximum/minimum degrees d_{MIN}, d_{MAX} (on the bottom), change as Algorithm 1 (green) and Algorithm 1 in [32] (blue) are executed. To ensure a fair comparison, the evolution of the algebraic connectivity is plotted against the cumulative number of edges that are either added or removed throughout the algorithms' operations, i.e., the number of operations required by the algorithm: the addition or removal of an edge is considered a single operation, whereas moving an edge is counted as two operations.

The simulations show that a network implementing Algorithm 1 can self-organize its graph into a 2-approximate k-regular graph with only about one-fifth the number of actions required by the algorithm presented in [32] to construct a k-regular graph. We further note that the self-organized graph generated by both algorithms increases its algebraic connectivity λ_2, approaching and then exceeding the lower bound $\lambda_{2,lb} := k - 2\sqrt{k-1} = 36$ for random k-regular graphs, see Proposition 1.

A. Open networks

We test the proposed Algorithm 1 in the context of open multi-agent systems, wherein agents are allowed to leave and join the network during the execution of the algorithm, see for instance [26], [33]. We consider a network with $n = 1000$ agents executing Algorithm 1, whose initial interaction pattern is already described by a random k-regular graph with $k = 50$. During the execution of the algorithm, agents are selected and removed from the network, thus simulating DoS attacks.

In the field of network security, to disrupt the operation of a network, attackers attempt to identify and remove key nodes. The common assumption here is that the attacker has comprehensive insight into the entire network structure and the ability to accurately identify the nodes whose removal would severely impact the connectivity and resilience of the network. To simulate such strategies, we consider the eigenvectors of the Laplace matrix associated with the algebraic connectivity, also known as Fiedler eigenvector. Each entry in the eigenvector corresponds to a node of the graph, with its sign indicating which part the node belongs to when the network is segmented into two parts based on the Fiedler vector. Attackers typically target and remove the nodes corresponding to the smallest absolute values in the Fiedler vector, as these nodes often occupy "central" positions in the graph. The removal of these kind of nodes can split the network in half, thereby may achieving the attacker's goal.

The aim of the simulation is that of showing that the network is able to self-organize itself when some agents are disconnected from the network, maintaining a high level of connectivity and keeping the number of edges limited to a maximum of $\frac{(k+2)n}{2}$. We terminate the simulation either when the graph becomes disconnected (i.e., the attack succeeds) or the number of remaining nodes is equal to $n = k + 1$ (i.e., the attack fails). To this aim, Figure 5 compares the evolution of λ_2 in two scenarios, one accord-

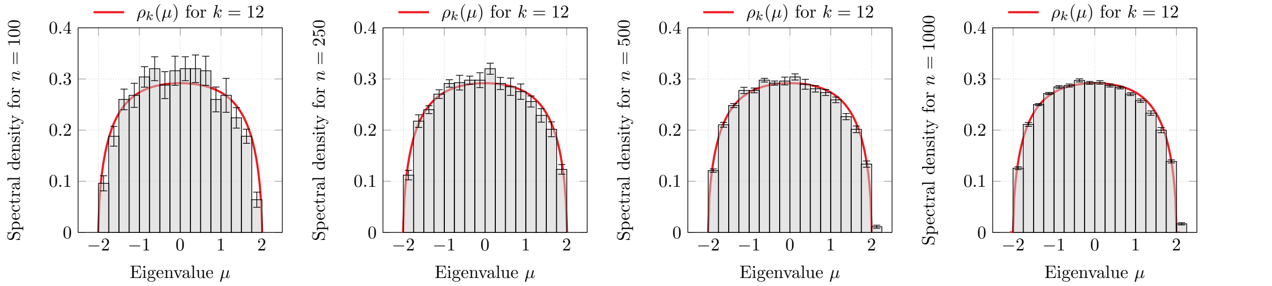

Fig. 2: Empirical eigenvalue density histogram of the normalized adjacency matrix $A_n/\sqrt{k-1}$ of graphs generated by Algorithm 1 in networks with an increasing number of agents $n \in \{100, 250, 500, 1000\}$ and fixed degree of regularity $k = 12$. The red curve represents the density ρ_k expected for large k-regular graphs (see Proposition 2).

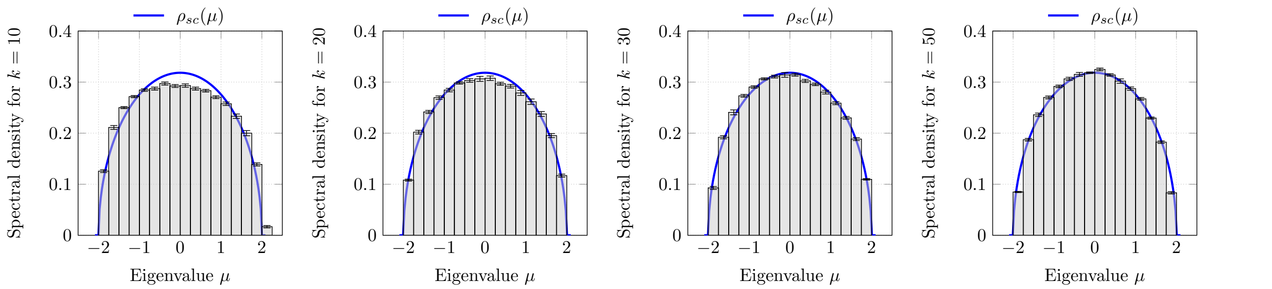

Fig. 3: Empirical eigenvalue density histogram of the normalized adjacency matrix $A_n/\sqrt{k - k^2/n}$ of graphs generated by Algorithm 1 in networks with $n = 1000$ agents and an increasing degree of regularity $k \in \{10, 20, 30, 50\}$. The blue curve represents the semicircle density ρ_{sc} expected for large k-regular graphs as $k \to \infty$ (see Proposition 3).

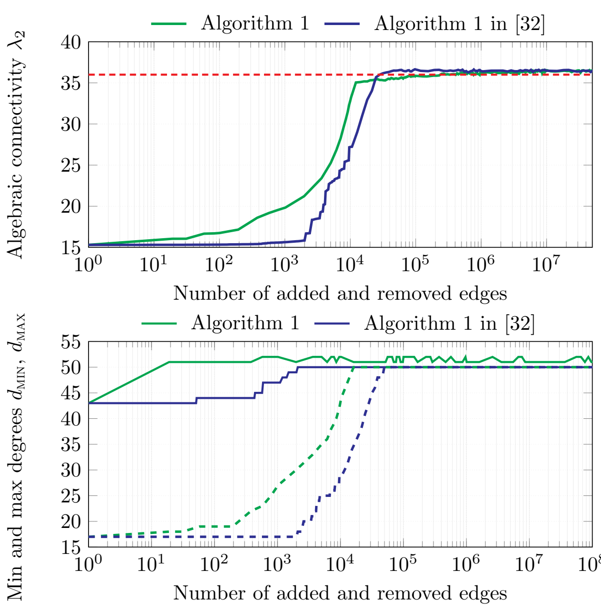

Fig. 4: Evolution of λ_2 (top) and the maximum/minimum degrees (bottom) against the number of added and removed edges during the execution of Algorithm 1 (green curves) and algorithm in [32] (blue curves).

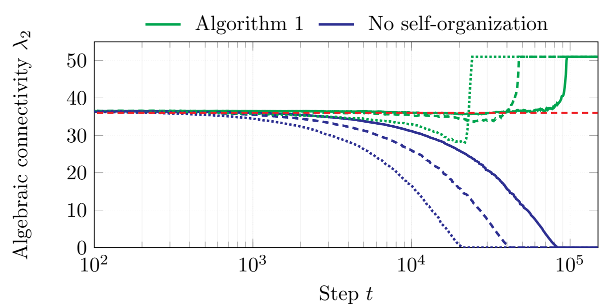

Fig. 5: Evolution of λ_2 in initially 50-regular graphs of $n = 1000$ nodes from which one node is disconnected every 100 steps (solid curve), 50 steps (dashed curve), 25 steps (dotted curve).

ing to Algorithm 1 (green curve) and the other when no self-organization of the graph is performed (blue curve). Moreover, we explore the network's resilience by varying the frequency of simulated attacks, namely every 100 steps (solid curve), every 50 steps (dashed curve), and every 25 steps (dotted curve).

Figure 5 clearly shows a significant decrease in algebraic connectivity when no self-organization of the graph is performed (blue curve), dropping from a value of about 36 to a value of 30 when only 120 nodes have been attacked and removed from the network, decreasing further until the network gets eventually disconnected into two components of approximately 80 nodes each. On the other hand, when executing Algorithm 1, the algebraic connectivity is kept high even when the attacks are very frequent. Remarkably, when the attacker is able to disrupt all nodes but $k + 1$ the graph is maintained connected with a complete topology and an algebraic connectivity equal to $\lambda_2 = k + 1 = 51$. These results emphasize the effectiveness of Algorithm 1 in robustifying the network's structure, ensuring it remains connected even when under DoS attacks.

V. Conclusions

We present a novel distributed algorithm for self-organizing any connected graph into a random 2-approximate k-regular graph, where k is a design parameter known by each agent. The algorithm is designed to be executed in open multi-agent networks, where nodes can join and leave the network at will. Numerical simulations demonstrate the effectiveness of the protocol by showcasing that the graphs produced are a close approximation to random k-regular graphs in terms of algebraic connectivity and spectral distribution. We also numerically compare our approach to existing techniques and highlighted the increased resilience to network disconnections caused by the loss of nodes or links. Future work will provide a formal proof that the proposed algorithm can maintain connectivity, and the spectral distribution of the obtained graphs is close to that of a random k-regular graph.

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Appendix A: Proof Sketch of Theorem 1

We present here only a proof sketch due to space constraints. First, we study the behavior of the minimum and maximum degrees while the agents execute Algorithm 1. For the minimum degree d_{MIN}, analyzing rules (A), (R), and (M) ensures that d_{MIN} does not decrease if $d_{\text{MIN}} \leq k$. Specifically, no rule decreases the degree of the agent with the minimum degree. If $d_{\text{MIN}} < k$, rule (A) will eventually increase d_{MIN} to at least k. For the maximum degree d_{MAX}, if $d_{\text{MAX}} \geq k{+}2$ and assuming $d_{\text{MIN}} \geq k$, the rules prevent d_{MAX} from increasing. If $d_{\text{MAX}} > k+2$, d_{MAX} eventually decreases to less than or equal to $k+2$ and remains there by executing rules (R) and (M). Under the assumption that $\mathcal{G}$ is initially connected and remains connected thereafter, Algorithm 1 ensures d_{MIN} converges to $[k, \infty)$ and d_{MAX} converges to $[k, k+2]$. This means the degree d_i of each agent $i \in \mathcal{V}$ converges to the interval $d_i \in [k, k{+}2]$, resulting in a 2-approximate k-regular graph.